Nature of solutions of differential equations associated with a class of one-dimensional maps

2004 J. Phys. A: Math. Gen. 37 11373

(http://iopscience.iop.org/0305-4470/37/47/006)

J. Phys. A: Math. Gen. **37** (2004) 11373–11377 PII: S0305-4470(04)85475-1

Nature of solutions of differential equations associated with a class of one-dimensional maps

M C Valsakumar[1], **A Rajan Nambiar**[2] **and P Rameshan**[3]

[1] Materials Science Division, IGCAR, Kalpakkam-603102, India
[2] Department of Physics, Government Arts & Science College, Kozhikode-673018, India
[3] Department of Physics, University of Calicut, Calicut University PO, 673635, India

E-mail: valsa@igcar.ernet.in

Received 25 August 2004
Published 10 November 2004
Online at stacks.iop.org/JPhysA/37/11373
doi:10.1088/0305-4470/37/47/006

Abstract
Discrete time evolution of one-dimensional maps is embedded in continuous time by truncating the Taylor series expansion of the time evolution operator to a finite order N. The fixed points of the ordinary differential equations thus obtained are unstable whenever $N > 4$ regardless of the details of the underlying map, so long as it is continuous and differentiable. Generalization of the truncated equations with $N = 3$ and 4 shows dynamical behaviour characteristic of systems with a riddled parameter space.

PACS numbers: 05.45.Pq, 02.30.Hq

1. Introduction

Time development of systems evolving continuously in time is modelled with differential equations, whereas difference equations (maps) are used to describe systems that evolve in discrete time. Here, we try to study the connection, if any, between solutions of discrete maps and a set of ordinary differential equations that are *derived* from the map using an ansatz described below. Basically, we truncate the Taylor series expansion of the time evolution operator corresponding to the discrete map at a finite order N to get an ordinary differential equation (ODE) of order N. Contrary to the expectation that the solution of such an ODE should converge to that of the discrete map (when $t =$ an integer) in the limit $N \to \infty$ we find surprising results. Our numerical studies of the ODEs thus obtained for a few maps (logistic map, tent map, etc), using adaptive fourth-order Runge–Kutta method, showed that the solutions became unbounded in a finite time for all the randomly chosen initial conditions (that we used) whenever $N \geqslant 5$ and finite. We report the results of our analysis prompted by this intriguing observation. In particular, we show that the fixed points of all these ODEs with $N \geqslant 5$ are unstable regardless of the details of the underlying map, so long as it is continuous and differentiable. Results of our ongoing study of the nature of solutions of these ODEs for arbitrary initial conditions will be reported elsewhere.

The nature of the solutions for truncation at $N \leqslant 4$ do depend on the details of the map. The ODEs with $N = 1$ and 2 show regular behaviour. The third- and fourth-order ODEs obtained for the logistic map show period doubling bifurcations leading to chaos, reverse bifurcations, instability in some regions etc. Unlike the logistic map, these equations show sudden switching from regular to chaos or to unbounded behaviour by a slight variation in the parameter, which is an indication of structural instability that was discussed in the context of systems with a riddled parameter space (Cazelles 2001, Kapitaniak *et al* 2000, Kim and Lim 2001, Krawiecki and Matyjaskiewicz 2001, Lai and Winslow 1994, Lai 2000, Lai and Andrade 2001, Medvinsky *et al* 2001, Maistrenko *et al* 1999, Woltering and Markus 2000, Yang 2000, 2001).

2. Ordinary differential equations corresponding to discrete maps

Consider a general one-dimensional map

$$x_{n+1} = f(x_n). \tag{1}$$

This discrete time evolution can be embedded in continuous time t by considering the equation

$$\hat{T}x(t) = f(x(t)) \tag{2}$$

where $\hat{T}$ is the unit time evolution operator

$$\hat{T} = \exp\left(\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{3}$$

We consider the nature of solutions of the sequence of ordinary differential equations

$$\sum_{j=0}^{N}\left(\frac{1}{j!}\right)\left(\frac{\mathrm{d}^j}{\mathrm{d}t^j}\right)x(t) = f(x(t)) \tag{4}$$

obtained by truncating the formal power series expansion of the time evolution operator $\hat{T}$ to order N.

3. Nature of solutions for $N \geqslant 5$

As mentioned in the introduction, one of the objectives of this paper is to examine the nature of solutions of the ordinary differential equations (4) for various values of N. For any map $f(x)$ that is continuous and differentiable, we prove that the fixed points of the ODEs represented by equation (4) are unbounded if $N \geqslant 5$.

For this purpose, we now consider the solutions of equation (4) with the initial condition $x(0) = x_\star + \epsilon$, where $x_\star$ is a fixed point ($f(x_\star) = x_\star$) of the map and ϵ is an infinitesimal.

Writing $x(t) = x_\star + \delta x(t)$ so that $\delta x(0) = \epsilon$, and using the relations

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}x(t) = \frac{\mathrm{d}^j}{\mathrm{d}t^j}(x_\star + \delta x(t)) = \frac{\mathrm{d}^j}{\mathrm{d}t^j}\delta x(t), \qquad \text{for} \quad j \geqslant 1,$$

and

$$x(t) - f(x(t)) = x_\star - f(x_\star) + \left(1 - \frac{\mathrm{d}}{\mathrm{d}x}f(x)_{|x=x_\star}\right)\delta x(t) + O(\delta x(t)^2),$$

one gets

$$\sum_{j=1}^{N}\left(\frac{1}{j!}\right)\left(\frac{\mathrm{d}^j}{\mathrm{d}t^j}\right)\delta x(t)+\alpha(x_\star)\delta x(t)=0 \tag{5}$$

to first order in $\delta x(t)$. In equation (5), $\alpha(x_\star) = 1 - \frac{\mathrm{d}}{\mathrm{d}x} f(x)_{|x=x_\star}$, and we have also used the fact that $x_\star - f(x_\star) = 0$. Since the above equation is linear we look for a solution of the form

$$\delta x(t) = c\exp(\mu t) \tag{6}$$

to get

$$\sum_{j=1}^{N}\left(\frac{1}{j!}\right)\mu^j+\alpha(x_\star)=0. \tag{7}$$

We now show that there will exist at least one μ with positive real part irrespective of the value of $\alpha(x_\star)$, if $N \geqslant 5$. This would imply that the fixed points of equation (4) are unstable for any finite $N \geqslant 5$ regardless of the specific functional form of $f(x)$.

Now, equation (7) is of the form

$$a_0\mu^N + a_1\mu^{N-1} + \cdots + a_{N-1}\mu + a_N = 0, \tag{8}$$

with $a_j = \frac{1}{(N-j)!}$ for $0 \leqslant j \leqslant (N-1)$ and $a_N = \alpha(x_\star)$. Nature of zeros of equation (8) can be examined by using the well-known Routh–Hurwitz theorem (Korn and Korn 1961). Define

$$U_0 = a_0, \qquad U_1 = a_1, \qquad U_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix},$$
$$U_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \qquad U_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}, \text{ etc.} \tag{9}$$

Then Routh–Hurwitz theorem states that the number of roots of equation (8) with positive real parts is equal to the number of sign changes in the sequence $\{U_j\}$. If at least one sign change occurs in the sequence $\{U_j\}$, then equation (8) has at least one root (μ) with positive real part and hence the solution of equation (4) is unstable in the neighbourhood of the fixed point $x_\star$.

We now calculate the sequence $\{U_j\}$ when $N > 5$. Substituting for a_j in equation (9), and on simplifying, we get

$$U_0 = \frac{1}{N!}, \qquad U_1 = \frac{1}{(N-1)!}, \qquad U_2 = \frac{2}{N!(N-2)!}, \tag{10}$$

which are positive for all positive N and

$$U_3 = -\frac{2(N-5)}{N!(N-1)!(N-3)!} \tag{11}$$

which is negative for all $N > 5$. Since there is at least one sign change in the sequence $\{U_j\}$, it follows that the fixed points of equation (4) are unstable for $N > 5$ irrespective of the details of the mapping function $f(x)$, so long as it is continuous and differentiable.

We now turn to the case $N = 5$. Explicit calculation shows that

$$U_0 = \frac{1}{5!}, \qquad U_1 = \frac{1}{4!}, \qquad U_2 = \frac{2}{5!3!}, \qquad U_3 = \frac{\alpha(x_\star)-1}{5!4!}$$

and

$$U_4 = -\frac{(\alpha(x_\star)-5/3)^2+20/9}{5!^2}. \tag{12}$$

From equation (12), it is clear that there is a sign change at U_3 and no further sign change at U_4, if $\alpha(x_\star) < 1$. On the other hand, if $\alpha(x_\star) > 1$, $U_3 > 0$ and $U_4 < 0$. Thus, there is at least one sign change in the sequence $\{U_j\}$, regardless of the details of the mapping function $f(x)$ for $N = 4$, as well. No such general conclusions can be drawn for the cases $N = 1, 2$, 3 and 4.

We now turn to the specific case of truncation of chaotic maps. Since a minimum of three dimensions is required for occurrence of chaos in ordinary differential equations, truncations to order $N = 1$ and 2 are uninteresting from this perspective. We also now know that the truncations to order $N \geqslant 5$ are also uninteresting as they lead to unbounded solutions. Therefore, in what follows, we concentrate on truncations with $N = 3$ and 4. It is found (Rajan Nambiar 2003) that the solutions of these equations show many features which are not shared by the solutions of the original discrete map. For concreteness, we present the results for the logistic map $f(x) = px(1 - x)$, $x \in [0, 1]$.

4. Truncation at N = 3 and 4

For this case, equation (4) becomes

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + 3\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 6(x - f(x)) = 0. \tag{13}$$

Linear stability analysis of the above equation, with $f(x) = px(1 - x)$ shows that both fixed points (0 and $1 - 1/p$) are stable for $p < 4$. Thus, the truncation at $N = 3$ leads to regular behaviour for all $p \in [0, 4]$, the range of p for which the logistic map is map of the unit interval to itself. This has to be contrasted with the occurrence of chaos in the logistic map when $p \in [3.66, \ldots, 4]$. However, the system of equations (13) do show chaotic behaviour when $p > 4$.

Using the scaled variables $X = (2p/9)x$ and $\tau = t/3$, equation (13) can be rewritten as

$$\frac{\mathrm{d}^3X}{\mathrm{d}\tau^3} + \frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} + \nu\frac{\mathrm{d}X}{\mathrm{d}\tau} + -\lambda X + X^2 = 0 \tag{14}$$

where $\nu = 2/3$ and $\lambda = 2(p - 1)/9$. In fact, the generalization of equation (14) obtained by allowing ν to take arbitrary values is equivalent to the equations studied by Coulett *et al* (1979) and Arneodo *et al* (1985) which exhibited striking features in its solution. The model shows regular, chaotic and or unstable behaviour for certain choices of the parameters ν and λ. For example, for a fixed ν, as λ is increased, we observe sequences of either finite or infinite number of period doubling bifurcations and reverse bifurcations with unstable regions in between. The bifurcation diagram is substantially different for a neighbouring ν. Thus, the significant aspect of this equation is that the nature of solution changes drastically for very small changes in the values of the parameters. It is as if the parameter space is riddled. The solution of ODE obtained by truncating at $N = 4$ also shows similar features.

5. Conclusions

The present work shows that the nature of solutions of the ordinary differential equations, obtained by truncating the power series of the time evolution operator corresponding to discrete maps, is very different from those of the discrete maps. In particular, the fixed points of the ODEs with $N \geqslant 5$ are unstable regardless of the details of the map. In contrast, the stability properties of the ODEs with $N \leqslant 4$ do depend on the details of the map. For the specific case of the logistic map, truncations at $N = 3$ and 4 lead to solutions characteristic of systems with a riddled parameter space. Study of the nature of solutions of the equations with $N \geqslant 5$ for arbitrary initial conditions would, indeed, be interesting.

Acknowledgments

ARN is grateful to UGC and Government of Kerala for granting him leave under FIP for this study.

References

Arneodo A, Coulett P H, Spiegel E A and Tresser C 1985 *Physica* D **14** 327
Cazelles B 2001 *Phys. Rev.* E **64** 032901
Coulett P, Tresser C and Arneodo A 1979 *Phys. Lett.* A **72** 268
Kapitaniak T, Maistrenko Y and Popovych S 2000 *Phys. Rev.* E **62** 1972
Kim S T and Lim W 2001 *Phys. Rev.* E **63** 026217
Korn G A and Korn T M 1961 *Mathematical Handbook for Scientists and Engineers* (New York: McGraw-Hill)
Krawiecki A and Matyjaskiewicz S 2001 *Phys. Rev.* E **64** 036216
Lai Y C 2000 *Phys. Rev.* E **62** R4505
Lai Y C and Andrade V 2001 *Phys. Rev.* E **64** 056228
Lai Y C and Winslow R L 1994 *Phys. Rev. Lett.* **72** 1640
Maistrenko Yu L, Maistrenko V L, Popovych O and Mosekilde E 1999 *Phys. Rev.* E **60** 2817
Medvinsky A B *et al* 2001 *Phys. Rev.* E **64** 021915
Rajan Nambiar A 2003 *PhD Thesis* University of Calicut
Woltering M and Markus M 2000 *Phys. Rev. Lett.* **84** 630
Yang H L 2000 *Phys. Rev.* E **62** R4509
Yang H L 2001 *Phys. Rev.* E **63** 036208